Supplement of

Modeling canopy-induced turbulence in the Earth system: a unified parameterization of turbulent exchange within plant canopies and the roughness sublayer (CLM-ml v0)

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S1 Numerical solution of Eqs. (16) and (17)

Richtmyer and Morton (1967, pp. 275–278) provide a numerical solution for Eqs. (16) and (17), common to that used for tridiagonal equations. These equations are

\begin{align*}
a_{i,j} \theta_{i-1}^{n+1} + b_{1i,j} \theta_{i}^{n+1} + b_{12,i} q_{i}^{n+1} + c_{1,i} \theta_{i+1}^{n+1} &= d_{i,j} \\ a_{2,i} q_{i-1}^{n+1} + b_{21,i} \theta_{i}^{n+1} + b_{22,i} q_{i}^{n+1} + c_{2,i} q_{i+1}^{n+1} &= d_{2,i} 
\end{align*}

(S1) (S2)

The solution involves rewriting these in the form

\begin{align*}
\theta_{i}^{n+1} &= f_{i,j} - e_{1i,j} \theta_{i+1}^{n+1} - e_{12,i} q_{i}^{n+1} \\
q_{i}^{n+1} &= f_{2,i} - e_{21,i} \theta_{i+1}^{n+1} - e_{22,i} q_{i+1}^{n+1}
\end{align*}

(S3) (S4)

Here, \( e \) is a \( 2 \times 2 \) matrix at each level \( i \), and \( f \) is a \( 2 \times 1 \) matrix at each level. These are found by substituting

\begin{align*}
\theta_{i}^{n+1} &= f_{i,j} - e_{1i,j} \theta_{i+1}^{n+1} - e_{12,i} q_{i}^{n+1} \\
q_{i}^{n+1} &= f_{2,i} - e_{21,i} \theta_{i+1}^{n+1} - e_{22,i} q_{i+1}^{n+1}
\end{align*}

(S5) (S6)

into Eqs. (S1) and (S2) to eliminate \( \theta_{i-1}^{n+1} \) and \( q_{i-1}^{n+1} \), and then substituting the resulting equation for \( \theta_{i}^{n+1} \) into that for \( q_{i}^{n+1} \) and vice versa. This gives

\begin{align*}
e_{11,i} &= c_{i,j} \left( b_{22,i} - a_{2,i} e_{22,i-1} \right) / \det \\
e_{12,i} &= -c_{2,i} \left( b_{12,i} - a_{1,i} e_{12,i-1} \right) / \det \\
e_{21,i} &= -c_{i,j} \left( b_{21,i} - a_{2,i} e_{21,i-1} \right) / \det \\
e_{22,i} &= c_{2,i} \left( b_{11,i} - a_{1,i} e_{11,i-1} \right) / \det
\end{align*}

(S7)

and
\[ f_{1,j} = \frac{(b_{22,j} - a_{2,j}e_{22,j-1})(d_{1,j} - a_{1,j}f_{1,j-1}) - (b_{12,j} - a_{1,j}e_{12,j-1})(d_{2,j} - a_{2,j}f_{2,j-1})}{\text{det}} \]
\[ f_{2,j} = \frac{-(b_{21,j} - a_{2,j}e_{21,j-1})(d_{1,j} - a_{1,j}f_{1,j-1}) + (b_{11,j} - a_{1,j}e_{11,j-1})(d_{2,j} - a_{2,j}f_{2,j-1})}{\text{det}} \]  
(S8)

with

\[ \text{det} = (b_{11,j} - a_{1,j}e_{11,j-1})(b_{22,j} - a_{2,j}e_{22,j-1}) - (b_{12,j} - a_{1,j}e_{12,j-1})(b_{21,j} - a_{2,j}e_{21,j-1}) \]  
(S9)

The \( e \) and \( f \) matrices are found sequentially upward through the canopy from \( i = 1 \) to \( N \) with
\[ e_{1,0} = e_{2,0} = e_{12,0} = e_{22,0} = 0 \] and \( f_{1,0} = f_{2,0} = 0 \). Then, \( \theta_i^{n+1} \) and \( q_i^{n+1} \) are calculated downward through the canopy from \( i = N - 1 \) to 1 using Eqs. (S3) and (S4) with \( \theta_N^{n+1} = f_{1,N} \) and \( q_N^{n+1} = f_{2,N} \).

### S2 Algebraic coefficients for Eqs. (16) and (17)

In the equations that follow, \( g_{\text{sun}}^{\text{sun}} = 2 \bar{g}_{b,j} \Delta L_{\text{sun},i} \) and \( g_{\text{sha}}^{\text{sha}} = 2 \bar{g}_{b,j} \Delta L_{\text{sha},i} \) are sunlit and shaded leaf conductances for sensible heat scaled to the canopy. \( g_{\text{sun}}^{\text{sun}} = g_{\text{sun},i} \Delta L_{\text{sun},i} \) and \( g_{\text{sha}}^{\text{sha}} = g_{\text{sha},i} \Delta L_{\text{sha},i} \) are similar conductances for evapotranspiration. The coefficients in Eqs. (16) and (17) are

\[ a_{i,j} = -g_{a,j-1} \]  
(S10)

\[ b_{1,i} = \frac{\rho_m \Delta z_i}{\Delta t} + g_{a,i-1} + g_{a,i} + g_{\text{sun}}^{\text{sun}}(1-\alpha_i^{\text{sun}}) + g_{\text{sha}}^{\text{sha}}(1-\alpha_i^{\text{sha}}) \]  
(S11)

\[ b_{2,i} = -g_{H,i}^{\text{sun}} \rho_i^{\text{sun}} - g_{H,i}^{\text{sha}} \rho_i^{\text{sha}} \]  
(S12)

\[ c_{i,j} = -g_{a,i} \]  
(S13)

\[ d_{i,j} = \frac{\rho_m \Delta z_i}{\Delta t} \theta_i^n + g_{H,i}^{\text{sun}} \theta_i^{\text{sun}} + g_{H,i}^{\text{sha}} \theta_i^{\text{sha}} \]  
(S14)

for temperature, and

\[ a_{2,j} = -g_{a,j-1} \]  
(S15)
\[ b_{21,j} = -g_{E,j} \gamma_i \alpha_i^{\text{sun}} - g_{E,j} \gamma_i \alpha_i^{\text{sha}} \]  
\[ (S16) \]

\[ b_{22,j} = \frac{\rho_m \Delta z_i}{\Delta t} + g_{a,i-1} + g_{a,j} + g_{E,j}^\text{sun} \left(1 - s_i^{\text{sun}} \beta_i^{\text{sun}}\right) + g_{E,j}^\text{sha} \left(1 - s_i^{\text{sha}} \beta_i^{\text{sha}}\right) \]  
\[ (S17) \]

\[ c_{2,j} = -g_{a,j} \]  
\[ (S18) \]

\[ d_{2,j} = \frac{\rho_m \Delta z_i}{\Delta t} q_i^n + g_{E,j}^\text{sun} \left[ q_{\text{sat}} \left(T^n_{(\text{sun},i)}\right) + s_i^{\text{sun}} \left(\delta_i^{\text{sun}} - T^n_{(\text{sun},i)}\right)\right] + g_{E,j}^\text{sha} \left[q_{\text{sat}} \left(T^n_{(\text{sha},i)}\right) + s_i^{\text{sha}} \left(\delta_i^{\text{sha}} - T^n_{(\text{sha},i)}\right)\right] \]  
\[ (S19) \]

for water vapor.

Special boundary conditions are needed at the top layer \( i = N \), where \( \theta_{i+1}^{n+1} = \theta_{\text{ref}}^{n+1} \) and \( q_{i+1}^{n+1} = q_{\text{ref}}^{n+1} \) so that

\[ c_{1,i} = 0 \]  
\[ (S20) \]

\[ d_{1,i} = \frac{\rho_m \Delta z_i}{\Delta t} \theta_i^n + g_{H,i}^\text{sun} \delta_i^{\text{sun}} + g_{H,i}^\text{sha} \delta_i^{\text{sha}} + g_{a,i} \theta_{\text{ref}}^{n+1} \]  
\[ (S21) \]

\[ c_{2,j} = 0 \]  
\[ (S22) \]

\[ d_{2,j} = \frac{\rho_m \Delta z_i}{\Delta t} q_i^n + g_{E,j}^\text{sun} \left[ q_{\text{sat}} \left(T^n_{(\text{sun},i)}\right) + s_i^{\text{sun}} \left(\delta_i^{\text{sun}} - T^n_{(\text{sun},i)}\right)\right] + g_{E,j}^\text{sha} \left[q_{\text{sat}} \left(T^n_{(\text{sha},i)}\right) + s_i^{\text{sha}} \left(\delta_i^{\text{sha}} - T^n_{(\text{sha},i)}\right)\right] + g_{a,j} q_{\text{ref}}^{n+1} \]  
\[ (S23) \]

and other terms are as given before.

Special boundary conditions are also needed for the first layer \( i = 1 \), where \( \theta_{i-1}^{n+1} = T_{0}^{n+1} \) and \( q_{i-1}^{n+1} = q_{0}^{n+1} \) are the ground surface temperature and water vapor concentration, respectively, so that

\[ a_{1,i} = 0 \]  
\[ (S24) \]

\[ b_{11,j} = \frac{\rho_m \Delta z_i}{\Delta t} + g_{a,i-1} + g_{a,i} + g_{H,i}^\text{sun} \left(1 - \alpha_i^{\text{sun}}\right) + g_{H,i}^\text{sha} \left(1 - \alpha_i^{\text{sha}}\right) - g_{a,i-1} \alpha_0 \]  
\[ (S25) \]
\[ b_{12,i} = -g_{H,i}^{\text{sun}} \beta_{i}^{\text{sun}} - g_{H,i}^{\text{sha}} \beta_{i}^{\text{sha}} - g_{a,i-1} \beta_{0} \]  
\( (S26) \)

\[ d_{1,i} = \frac{\rho_{m}\Delta z_{i}}{\Delta t} \theta_{i}^{n} + g_{H,i}^{\text{sun}} \delta_{i}^{\text{sun}} + g_{H,i}^{\text{sha}} \delta_{i}^{\text{sha}} + g_{a,i-1} \delta_{0} \]  
\( (S27) \)

\[ a_{2,i} = 0 \]  
\( (S28) \)

\[ b_{21,i} = -g_{E,i}^{\text{sun}} \alpha_{i}^{\text{sun}} - g_{E,i}^{\text{sha}} \alpha_{i}^{\text{sha}} - h_{s0}s_{0}g_{s0} \alpha_{0} \]  
\( (S29) \)

\[ b_{22,i} = \frac{\rho_{m}\Delta z_{i}}{\Delta t} + g_{a,i} + g_{E,i}^{\text{sun}} \left( 1-s_{i}^{\text{sun}} \beta_{i}^{\text{sun}} \right) + g_{E,i}^{\text{sha}} \left( 1-s_{i}^{\text{sha}} \beta_{i}^{\text{sha}} \right) - h_{s0}s_{0}g_{s0} \beta_{0} \]  
\( (S30) \)

\[ d_{2,i} = \frac{\rho_{m}\Delta z_{i}}{\Delta t} q_{i}^{n} + g_{E,i}^{\text{sun}} \left[ q_{\text{sat}}^{n} \left( T_{\text{sun},i}^{n} \right) + s_{i}^{\text{sun}} \left( \delta_{i}^{\text{sun}} - T_{\text{sun},i}^{n} \right) \right] + g_{E,i}^{\text{sha}} \left[ q_{\text{sat}}^{n} \left( T_{\text{sha},i}^{n} \right) + s_{i}^{\text{sha}} \left( \delta_{i}^{\text{sha}} - T_{\text{sha},i}^{n} \right) \right] \]  
\( + h_{s0} \left[ q_{\text{sat}}^{0} \left( T_{0}^{0} \right) + s_{0} \left( \delta_{0} - T_{0}^{0} \right) \right] g_{s0} \)  
\( (S31) \)

and other terms are as given before.